NCD-003-1162001 Seat No. ________

M. Sc. (Mathematics) (CBCS) (Sem. II) Examination

April / May – 2017

CMT-2001 : Algebra - II

(New Course)

Faculty Code : 003

Subject Code : 1162001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Each question carries equal marks.

(2) All questions are compulsory.

1 Answer following short questions : **7×2=14**

(i) Define R-submodule of an R-module M. Let $m \in M$ and M be an R-module. Prove that $R_x = \{sx \mid s \in R\}$ is an R-submodule of M.

(ii) Prove following elementary properties of an R-module M : (a) $0m = 0, \ \forall \ m \in M$ and (b) $a0 = a, \forall \ a \in R$.

(iii) Prove that $x^3 + 3x + 2 \in \mathbb{Z}_7[x]$ is an irreducible polynomial.

(iv) Let $f(x) = x^3 + 6x^2 + 7x + 8 \in \mathbb{Z}[x]$. Find $f(x-2)$ and using Eisenstein criterian prove that $f(x)$ is irreducible polynomial over $\mathbb{Z}[x]$.

(v) Define transcendental element over Q and give three transcendental elements of $\mathbb{R}$ over Q.

(vi) Write down minimal polynomial of $2^{1/4}$ over Q.

(vii) Prove or disprove $Q\left(2^{1/4}, i\right)/Q$ is a Galois extension.

2 Attempt any two : **2×7=14**

(a) Prove that every finite extension is always an algebraic extension.

(b) Define finite extension. Let $E|_F$, $K|_E$ be two finite extensions. Prove that $K|_F$ is also a finite extension.

(c) Prove or disprove $Q\left(\sqrt{2}, \sqrt{3}, ..., \sqrt{p}, ...\right)\Big|Q$ and $\bar{Q}|Q$ both are infinite algebraic extensions.

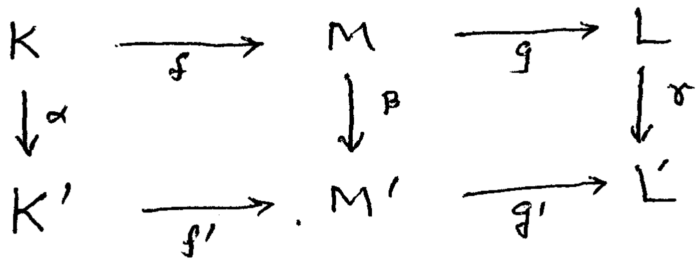
(d) Let F be a finite field. Prove that $F^* = F - \{0\}$ is a cyclic group under multiplication.

3 Attempt any one : **1×14=14**

(a) Let $E|_F$ and $K|_E$ be both finite seperable extensions. Prove that $K|_F$ is also a finite seperable extension.

(b) State and prove fundamental theorem of Galois theory.

(c) Define an exact infinite sequence of R-homomorphisms of R-modules. Suppose following diagram of R-modules and R-homomorphism is commutative and it has exact rows.



Prove that (i) β is one-one if α, r, f' all are one-one maps and (ii) β is onto if α, r, g all are onto maps.

4 Answer any two : **2×7=14**

(a) Let $(N_i)_{i \in \wedge}$ be a family of R-submodules of an R-module M. Prove that $\bigcap_{i \in \wedge} N_i$ is also an R-submodule of M.

(b) Let $f : M \to N$ be an R-homomorphism on R-modules. Prove that ker f and $f(M)$ are R-submodules of M and N respectively.

(c) Using Eisenstein criterian prove that $g(x) = 1 + x + x^2 + ... + x^{p-1}$ (p is prime) and $g(x+1)$ both are irreducible polynomials over $Q[x]$.

(d) Let F be a field. Prove that the prime subfield of F is either isomorphic to Q or it is isomorphic to $\mathbb{Z}_p$, for some prime p.

(e) Let R be a ring with unity. Prove that an R-module M is cyclic iff $M \simeq \frac{R}{I}$, for some left ideal I of R.

5 (a) Attempt any one of following : **1×8=8**

(1) For a field prove that following statements are equivalent :

$I \to K$ is algebraically closed.

$II \to$ if $p(x) \in K[x]$ and $p(x)$ is an irreducible polynomial then deg $p(x) = 1$.

$III \to$ For any $f(x) \in K[x]$ with deg $f(x) \geq 1$, $f(x)$ can be split into linear factors in $K[x]$.

$IV \to$ For any $f(x) \in K[x]$ with deg $f(x) \geq 1$, K contains all the roots of $f(x)$.

(2) State and prove primitive element theorem.

(b) Attempt any two of followings : **2×3=6**

(1) Define algebraically closed field. Which of followings is/are algebraically closed fields ?

Q, $\mathbb{R}$, $\mathbb{C}$, $\bar{Q}$.

(2) Prove that $F[x]$ is an F-module.

(3) Define F-automorphism for a field extension E/F and write down Galois group $G(\mathbb{C} \mid \mathbb{R})$ for the field extension $\mathbb{C} \mid \mathbb{R}$.
